4763. *Proposed by William Weakley.*

Let K be a field and let S be a nonempty subset of K that is closed under subtraction.

a) For all K and S, characterize the functions $f : S \to K$ such that

$$f(x)f(y) = f(x-y) \text{ for all } x, y \in S.$$

b) As K and S vary, what finite cardinalities can the set of such functions have?

We received 5 submissions out of which 4 were complete and correct. We present the solution by the Missouri State University Problem Solving Group, lightly edited.

a) Since S is nonempty and closed under subtraction, S is an additive subgroup of K. In particular, $0 \in S$. Let $x \in S$. Then

$$f(0) = f(x-x) = f(x)^2. \tag{1}$$

Letting $x = 0$ we see that $f(0) \in \{0, 1\}$. If $f(0) = 0$, then (1) implies $f(x) = 0$ for all $x \in S$. If $f(0) = 1$, then (1) implies $f(x) \in \{-1, 1\}$ for all $x \in S$. Therefore f is either identically 0 or it is a homomorphism from the additive group S to the multiplicative group $\{\pm 1\}$. Conversely, given such a homomorphism f and $x, y \in S$, the function f satisfies

$$f(x-y) = f(x)f(y)^{-1} = f(x)f(y),$$

since the elements of $\{\pm 1\}$ are their own inverses.

b) Let $Hom(S, \{\pm 1\})$ denote the set of all homomorphisms from S to $\{\pm 1\}$. Defining

$$(f \oplus g)(s) = f(s)g(s),$$
$$(r \odot f)(s) = f(s)^r$$

for $f, g \in Hom(S, \{\pm 1\})$, $r \in \mathbb{F}_2$, and $s \in S$ makes $Hom(S, \{\pm 1\})$ an $\mathbb{F}_2$-vector space. Therefore, if $Hom(S, \{\pm 1\})$ is finite, it contains 2^d elements where d is the dimension of the vector space.

To show that all nonnegative values of d are possible, first consider $d \geq 1$. Let $K = S = \mathbb{Q}[x]/p(x)$, where p is an irreducible polynomial of degree d. Then $S \cong \mathbb{Q}^d$ and $|Hom(S, \{\pm 1\})| = 2^d$, since there are two choices for each of the d basis vectors to map to.

For $d = 0$, consider $K = \mathbb{F}_2$. Then $Hom(S, \{\pm 1\}) = Hom(S, \{1\})$ and there is just one homomorphism.

Therefore, including the function that is identically zero, the possible finite values for the number of functions satisfying the conditions of the problem are $2^d + 1$ with $d \geq 0$.